Chapter 9

Algebra: Matrices, Vectors etc.

The human mind has never invented a labor-saving machine equal to algebra

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We now meet the ideas of *matrices* and *vectors.* While they may seem rather odd at first they are vital for studies in almost all subjects. The easiest way to see the power of the idea is to consider simultaneous equations. Suppose we have the set of equations

$$\begin{aligned} 3x - 5y &= 12 \\ x + 5y &= 24 \end{aligned}$$

We can find the solution $x = 9 \quad y = 3$ in several ways . For example if we add the second equation to the first we have equations

$$\begin{aligned} 4x &= 36 \\ x + 5y &= 24. \end{aligned}$$

Thus $x = 9$ and substituting 9 in the second equation gives $9 + 5y = 24$ or $5y = 15$ giving $y = 3$. Many mathematical models result in sets of simultaneous equations, like these except much more complex which need to be solved, or perhaps just to be examined. To do this more easily the matrix was invented. The essence of the set of equations

$$\begin{aligned} 3x - 5y &= 12 \\ x + 5y &= 24 \end{aligned}$$

is captured in the array or matrix of coefficients $\begin{pmatrix} 3 & -5 \\ 1 & 5 \end{pmatrix}$ or the *augmented matrix* $\begin{pmatrix} 3 & -5 & 12 \\ 1 & 5 & 0 \end{pmatrix}$ These arrays of numbers are called matrices. To save space we often give matrices names in boldface, for example

$$\mathbf{A} = \begin{pmatrix} 3 & -5 & 12 \\ 1 & 5 & 24 \\ 6 & 8 & -3 \\ 11 & 0 & 0 \end{pmatrix}$$

or

$$\mathbf{X} = \begin{pmatrix} 3 & -5 & 12 & 0 \\ 1 & 5 & 24 & 0 \end{pmatrix}.$$

We define an $r \times c$ matrix as a rectangular array of numbers with r rows and c columns, for example $\mathbf{A}$ above is a 4×3 matrix while $\mathbf{X}$ is 2×4 is

A matrix with just one column is called a *column vector* while one with just one row is a *row vector*, for example a column vector

$$\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

and a row vector

$$\mathbf{b} = \begin{pmatrix} 3 & -5 & 12 & -19 \end{pmatrix}$$

We use matrices in ways which keep our links with systems of equations. Before looking at the arithmetic of matrices we see how we can use them to come up with a general method of solving equations.

9.0.5 Equation Solving.

If you were to look at ways people use to solve equations you would be able do deduce some simple rules.

1. Equations can be multipled by a non-zero constant
2. Equations can be interchanged
3. Equations can be added or subtracted to other equations

If equations are manipulated following these rules they may look different but *they have the same solutions* as when you started. We can solve equations by writing the coefficients in the *augmented matrix* form and manipulating as follows

1. rows of the matrix may be interchanged
2. rows of the matrix may be multiplied by a nonzero constant.

3. rows can be added (or subtracted) to (from) other rows

Our aim is to reduce the matrix to what is known as *row echelon form.* This means that:

- the leading non zero term in each row is a one.
- Also the leading 1 in the first row lies to the left of that in the second row and so on. More precisely the leading 1 in any row lies to the left of the leading ones in all the rows below it.

For example

$$\begin{pmatrix} 1 & -5 & 12 \\ 0 & 0 & 24 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -5 & 12 \\ 0 & 1 & 24 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 12 & -19 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The reason for this will become apparent when we do it.
Lets try it out: We start with the equations

$$\begin{aligned} 2x + y + 2z &= 10 \\ x - 2y + 3z &= 2 \\ -x + y + z &= 0 \end{aligned}$$

in this case the coefficients are

$$\begin{pmatrix} 2 & 1 & 2 & 10 \\ 1 & -2 & 3 & 2 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

We are allowed to manipulate rows, these are *row operations*, to try and get to the row echelon form. Thus we have

1. Add row 2 to row 3 to get $\begin{pmatrix} 2 & 1 & 2 & 10 \\ 1 & -2 & 3 & 2 \\ 0 & -1 & 4 & 2 \end{pmatrix}$
2. Subtract row 2 from row 1 to get $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 1 & -2 & 3 & 2 \\ 0 & -1 & 4 & 2 \end{pmatrix}$
3. Subtract row 1 from row 2 $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & -5 & 4 & -6 \\ 0 & -1 & 4 & 2 \end{pmatrix}$

4. Tidy to get $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 5 & -4 & 6 \\ 0 & 1 & -4 & -2 \end{pmatrix}$

5. Subtract 5 times row 3 from row 2 to get $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 0 & 16 & 16 \\ 0 & 1 & -4 & -2 \end{pmatrix}$

6. Interchange rows 2 and 3 $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 16 & 16 \end{pmatrix}$

7. Tidy $\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

This seems more like row echelon .

This last matrix corresponds to the set of equations

$$\begin{aligned} x + 3y - z &= 8 \\ y - 4z &= -2 \\ z &= 1 \end{aligned}$$

These are much easier to solve! Here

$$z = 1 \quad y = 2 \quad x = 3.$$

It is often nicer to go a bit further and get rid of as much of the upper triangle as possible. Clearly the leading 1 in each row can be used to get zeros in the column above it. The resulting matrix is called *reduced row echelon form* of the original matrix. Here we get

$$\begin{pmatrix} 1 & 3 & -1 & 8 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

It does really matter a great deal to us which we use since we are only interested in solutions.

Lets look at another example

$$\begin{aligned} 6x + 3y + 6z &= 9 \\ x + 2y &= 16 \\ 4x + 5y + 1z &= 18 \end{aligned}$$

The augmented form is

$$\begin{pmatrix} 6 & 3 & 6 & 9 \\ 1 & 2 & 0 & 6 \\ 4 & 5 & 1 & 18 \end{pmatrix}$$

We have

$$\begin{pmatrix} 6 & 3 & 6 & 9 \\ 1 & 2 & 0 & 6 \\ 4 & 5 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & -9 & 6 & -27 \\ 1 & 2 & 0 & 6 \\ 0 & -3 & 1 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 3 & -2 & 9 \\ 0 & 3 & -1 & -6 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & -2/3 & 3 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

Some steps have been concatenated!

What can go wrong

In reality nothing much can go wrong but we need to examine a couple of cases where the results we obtain require some thought.

1. Suppose we end up with *a row of zeros*. This is no problem, except when the number of non-zero rows is less that the number of variables. This just means there is not an unique solution e.g

$$\begin{aligned} x + 2y - z &= 0 \\ x + z &= 3 \\ 2x + 2y &= 3 \end{aligned}$$

We have

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 2 & 2 & 0 & 3 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & -6 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This corresponds to

$$\begin{aligned} x + y + z &= -69 \\ y - z &= -3/2 \end{aligned}$$

Now there is a solution for these equations but it is not the explicit unique type we have been dealing with up to now. If z is known, say z_0 then it follows $x = 3 - z_0$ and $y = (2z_0 - 3)/2$. We have a solution for every z_0 value. Technically there are an *infinite number of solutions.* It is obvious if you think about it that if you have fewer equations than variables (unknowns) then you will not have a simple solution.

If we have 2 rows all zero then we have to give a value to two variables, if 3 then 3 variables and so on.

2. No Solution
Of course your equations may not have a solution in that they are contradictory, for example:

$$x = 1 \quad y = 3 \quad x = -2 \quad z = 16$$

We recognize the equations are contradictory (have no solutions at all) in the following way. *If we have a row of which is all zero except for the very last element then the equations have no solution.* For example:
Suppose we have the equations

$$\begin{aligned} x - 2y - 3z &= 1 \\ 2x + cy + 6z &= 6 \\ -x + 3y + (c-3)z &= 0 \end{aligned}$$

where c is some constant. We proceed to row echelon

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & c & 6 & 6 \\ -1 & 3 & c-3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & c+6 & 0 & 4 \\ 0 & 1 & c & 0 \end{pmatrix}$$

Before we go further what happens if $c = -6$? The middle row of our matrix corresponds to 0=4 which is nonsense. Thus the original equation set does not have a solution when $c = -6$

However we will just carry on

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & c & 6 & 6 \\ -1 & 3 & c-3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & c+6 & 2c & 6 \\ 0 & 1 & c & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2c-c(c+6) & 4 \\ 0 & 1 & c & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & c & 0 \\ 0 & 0 & 2c-c(c+6) & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & c & 0 \\ 0 & 0 & -4c-c^2 & 1 \end{pmatrix}$$

Now if $-4c - c^2 = 0$, that is $c = 0$, or $c = -4$ our last equation is $0 = 1$ which is clearly nonsense! This means that the original equations had no solution.

You may feel that this is a bit of a sledge hammer to crack a nut, but there is a real purpose to our exercise. If you move away from the trivial cases then the scheme we have outlined above is the best approach. It is also the technique use in the computer programs available for equation solving. In addition the shape of the reduced row echelon form tell us a lot about matrices. Often we have a system of equations where we have some parameters e.g. using our techniques above we can find the range of values, or perhaps the values themselves when solutions are possible.

The row elimination ideas we have outlined are known as *Gaussian elimination* in numerical circles. The algorithms which bear tis name, while very much slicker are based on these simple ideas.

Exercises

1. Solve

(a)

$$\begin{aligned} 2x + 3y &= 7 \\ 5x - y &= 9 \end{aligned}$$

(b)

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 5y + 7z &= 1 \\ -2x - 4y - 5z &= 1 \end{aligned}$$

(c)

$$\begin{aligned} v - w - x - y - z &= 1 \\ 2v - w + 3x + 4z &= 2 \\ 2v - 2w + 2x + y + z &= 1 \\ v + x + 2y + z &= 0 \end{aligned}$$

(d)

$$\begin{aligned} w + 2x - 3y - 4z &= 6 \\ w + 3x + y - 2z &= 4 \\ 2w + 5x - 2y - 5z &= 10 \end{aligned}$$

2. Consider the equations

$$\begin{aligned} v - w - x - y - z &= 1 \\ 2v - w + 3x + 4z &= 2 \\ 2v - 2w + 2x + y + z &= 1 \\ v + x + 2y + z &= c \end{aligned}$$

For what values of c do these equations have a unique solution? Are there any values of c for which there is no solution?

9.0.6 More on Matrices

If we have an $n \times m$ matrix $\mathbf{A}$ we need some way of referring to a particular element. It is common to refer to the (ij)th element meaning the element in row i and column j. We think of the matrix as having the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} & a_{3n} \\ cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{m,n-1} & a_{m,n} \end{pmatrix}$$

If we have a typical ijth element we sometimes write

$$\mathbf{A} = (a_{ij})$$

The *unit matrix* is an $n \times n$ matrix with ones on the diagonal and zeros elsewhere, usually written $\mathbf{I}$ for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So $\mathbf{A}$ is a unit matrix if

1. It is square.
2. The elements a_{ij} satisfy $a_{ii} = 1$ for all i and $a_{ij} = 0$ for all $i \neq j$

9.0.7 Addition and Subtraction

We can add or subtract matrices *that have the same dimensions* by just adding or subtracting the corresponding elements. For example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$$

when $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & -3 \\ -2 & -1 \\ 0 & 5 \end{pmatrix}$ then $\mathbf{A}+\mathbf{B} = \begin{pmatrix} -3 & -1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}$

while $\mathbf{A}-\mathbf{B} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \\ 4 & 5 \end{pmatrix}$

Multiplication by a scalar (number)

We can multiply a matrix $\mathbf{A}$ by a number s to give $s\mathbf{A}$ which is the matrix whose elements are those of $\mathbf{A}$ multiplied by s, so if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{m,n-1} & a_{m,n} \end{pmatrix}$$

then

$$s\mathbf{A} = \begin{pmatrix} sa_{11} & sa_{12} & \cdots & sa_{1,n-1} & sa_{1n} \\ sa_{21} & sa_{22} & \ldots & sa_{2,n-1} & sa_{2n} \\ sa_{31} & sa_{32} & \cdots & sa_{3,n-1} & sa_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ sa_{m1} & sa_{m2} & \cdots & sa_{m,n-1} & sa_{m,n} \end{pmatrix}$$

We use the term *scalar* for quantities that are not vectors.

Transpose of a matrix

If we take a matrix $\mathbf{A}$ and write the columns as rows then the new matrix is called the transpose $\mathbf{A}$ written $\mathbf{A}^{\mathsf{T}}$ or $\mathbf{A}'$

Thus if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 11 & 12 & 0 \end{pmatrix}$ then $\mathbf{A}^{\mathsf{T}} = \begin{pmatrix} 1 & 11 \\ 2 & 12 \\ 4 & 5 \end{pmatrix}$ Notice that $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$.

Any matrix that satisfies

$$\mathbf{A} = \mathbf{A}^{\mathsf{T}}$$

is said to be *symmetric*. If

$$\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$$

then it is anti-symmetric.

Multiplication of Matrices

This is a rather more complicated topic. We define multiplication in a rather complex way so that we keep a connection with systems of equations. Suppose $\mathbf{A}$ is an $n \times p$ matrix and $\mathbf{B}$ is a $p \times m$ matrix. Then the (ij)th element of $\mathbf{AB}$ is

$$\sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} + \ldots + a_{ip-1}b_{p-1j} + a_{ip}b_{pj}$$

Note that $\mathbf{AB}$ is an $n \times m$ matrix. One way of thinking of this is to notice that the (ij)th element of the product matrix is made up by multiplying elements in the ith row of the first matrix by the corresponding elements in the jth column of the second matrix. The products are then summed.

examples

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} = 1 \times 7 + 2 \times 6 + 3 \times 4 = 31$$

$$\begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 14 & 21 \\ 6 & 12 & 18 \\ 4 & 8 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 12 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 22 \\ 28 & 124 \end{pmatrix}$$

Some consequences are

- You can only multiply matrices if they have the right dimensions.
- In general $\mathbf{AB} \neq \mathbf{BA}$
- $\mathbf{AI} = \mathbf{A}$
- $\mathbf{IA} = \mathbf{A}$ but $\mathbf{I}$ has different dimensions to that above
- $\mathbf{A0} = \mathbf{0}$
- $\mathbf{0A} = \mathbf{0}$ but $\mathbf{0}$, a matrix of zeros, has different dimensions to that above

As we said the reason for this strange idea is so that it ties in with linear equations, thus if

$$\begin{aligned} x + 2y &= u \\ 4x + 9y &= v \end{aligned}$$

and

$$\begin{aligned} v + 4y &= 3 \\ 2v - y &= 0 \end{aligned}$$

these can be written in matrix form

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{u}$$

and

$$\mathbf{B}\mathbf{u} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

So we can write both e can write systems of equations as one matrix equation

$$\mathbf{B}\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 17 & 38 \\ -2 & -5 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

This is exactly the same set of equations we would have had if we had eliminated u and v without any matrices.

Inverses

So we have a whole set of algebraic operations we can use to play with matrices, except we have not defined division since if we can multiply then why not divide?

For a (non-zero) number z we can define the inverse z^{-1} which satisfies

$$zz^{-1} = z^{-1}z = 1.$$

In the same way we say that the matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$. if there is a matrix $\mathbf{A}^{-1}$ which satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

Beware not all matrices have inverses! Those that do are said to be *non-singular* otherwise a matrix which does not have an inverse said to be is *singular*. If you think about it you will see that only square matrices can have inverses. Suppose $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is another $n \times n$ matrix. If

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

where I is an $n \times n$ unit matrix then $\mathbf{B}$ is the inverse of $\mathbf{A}$. Notice $\mathbf{A}$ must be square *but* not all square matrices have inverses.

We can of course find the inverse by solving equations. For example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we then solve the four equations .

$$\begin{aligned} ae+bg &= 1 \\ af+bh &= 0 \\ ce+dg &= 0 \\ cf+dh &= 1 \end{aligned}$$

Not a very promising approach. However we can use the row-echelon ideas to get an inverse. All we do is take a matrix $\mathbf{A}$ and paste next to it a unit matrix $\mathbf{I}$. Write this augmented matrix as $\mathbf{B} = (\mathbf{AI})$.

We row reduce $\mathbf{B}$ to *reduced row echelon form*. The position of the original $\mathbf{I}$ is the inverse. For example suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$ then

$$\mathbf{B} = (\mathbf{AI}) \begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{pmatrix}$$

We get using row operations

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -4 & 1 \end{pmatrix}$$

and the inverse is $\mathbf{A}^{-1} = \begin{pmatrix} 9 & 2 \\ -4 & 1 \end{pmatrix}$

Of course we check

$$\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} \begin{pmatrix} 9 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$$

What can go wrong?

1. If you manage to convert the left hand matrix $\mathbf{A}$ to a unit matrix $\mathbf{I}$ then you have succeeded.

2. Sometimes as you manipulate the augmented matrix $\mathbf{B}$ you introduce a row of zeros into the position where you placed $\mathbf{A}$. In this case you can stop as *there is no solution.*

Consider $\mathbf{A} = \begin{pmatrix} 6 & 3 & 6 \\ 1 & 2 & 0 \\ 4 & 5 & 0 \end{pmatrix}$. The augmented matrix is $\mathbf{B} = \begin{pmatrix} 6 & 3 & 6 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \end{pmatrix}$
Now using row operations we have

$$\begin{pmatrix} 6 & 3 & 6 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -9 & 6 & 1 & -6 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 3 & 1 & -6 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & -4 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1/3 & 0 & -4/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -2/9 & -3 & 4/3 \\ 0 & 1 & 0 & 1/9 & 2 & -1/3 \\ 0 & 0 & 1 & 1/3 & 2 & -1 \end{pmatrix}$$

giving us our inverse

$$\begin{pmatrix} -2/9 & -3 & 4/3 \\ 1/9 & 2 & -1/3 \\ 1/3 & 2 & 0 \end{pmatrix}$$

Consider now $\mathbf{A} = \begin{pmatrix} 6 & 3 & 6 \\ 1 & 2 & 0 \\ 4 & 8 & 0 \end{pmatrix}$. The augmented matrix is $\mathbf{B} = \begin{pmatrix} 6 & 3 & 6 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 1 & 0 \end{pmatrix}$
Now using row operations we have

$$\begin{pmatrix} 6 & 3 & 6 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -9 & 6 & 1 & -6 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Given the zeros we know there is no inverse!

Of course we can think of solving equations using inverse matrices. It is almost always better to use row operations on the augmented matrix but we can proceed as follows. If we have the equations

$$\begin{aligned} 6x + 3y + 6z &= 9 \\ x + 2y &= 6 \\ 4x + 5y + z &= 18 \end{aligned}$$

this can be written as

$$\begin{pmatrix} 6 & 3 & 6 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 18 \end{pmatrix}$$

so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & 3 & 6 \\ 1 & 2 & 0 \\ 4 & 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ 6 \\ 18 \end{pmatrix} = \begin{pmatrix} -2/9 & -3 & 4/3 \\ 1/9 & 2 & -2/3 \\ 1/3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 18 \end{pmatrix}$$

In general if

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

provided $\mathbf{A}^{-1}$ exists.

Summary

1. The transpose of $\mathbf{A}$ written $\mathbf{A}^\mathsf{T}$ is the matrix made by writing the rows of $\mathbf{A}$ as columns in $\mathbf{A}^\mathsf{T}$.

2. $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^\mathsf{T}$

3. The zero matrix is the $n \times m$ array of zeros e.g $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. The unit matrix $\mathbf{I}$ (of order n) is the $n \times m$ matrix with 1's on the diagonal and zeros elsewhere e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. The matrix $\mathbf{A}$ has an inverse $\mathbf{B}$ iff $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. $\mathbf{B}$ is written $\mathbf{A}^{-1}$.

6. A matrix which has an inverse is said to be non-singular.

7. Do remember that except in special cases $\mathbf{AB} \neq \mathbf{BA}$

Exercises

1. Given $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 0 \end{pmatrix}$ compute $\mathbf{AB}$ and $\mathbf{BA}$

2. Show that $\begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 3 \end{pmatrix}$ is skew symmetric.

3. If $\mathbf{A} = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & 0 \end{pmatrix}$ show that $\mathbf{A} = \mathbf{A}^2$

4. Show that $\mathbf{AB}^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$

5. Show that the inverse of $\mathbf{AB}$ is $\mathbf{B}^{-1}\mathbf{A}^{-1}$

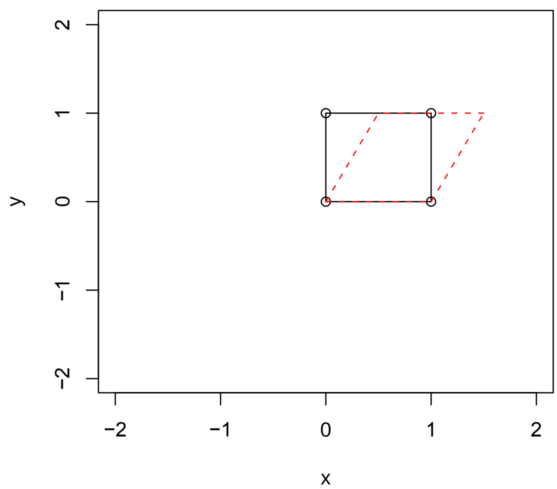
6. Find the inverse of $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 0 \end{pmatrix}$

Geometry

We write the point (x, y) in the plane as the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. If $\mathbf{A}$ is a 2×2 matrix $\mathbf{Ax}$ transforms $\mathbf{x}$ into a new point. Suppose $\mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$ Then

1. $\mathbf{A} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. $\mathbf{A} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
3. $\mathbf{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$
4. $\mathbf{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

If we plot the 4 points (0,0),(0,1),(1,1),(0,1) and their transforms we get



9.1 Determinants

Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can show that this has an inverse $\begin{pmatrix} e & f \\ g & d \end{pmatrix}$ when $\nabla = ad - bc \neq 0$, see 9.0.7. The quantity ∇ is called the *determinant* of the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and is written $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ or $\det(\mathbf{A})$. Similarly $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has an inverse when

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \neq 0$$

The general definition of a determinant of an $n \times n$ matrix $\mathbf{A}$ is as follows.

1. If $n = 1$ then $\det(\mathbf{A}) = a_{11}$
2. if $n > 1$ Let M_{ij} be the *determinant* of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ by deleting row i and column j. M_{ij} is called a *minor*.

Then

$$\det(\mathbf{A}) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14} + \ldots (-1)^{n+1}a_{1n}M_{1n} = \sum_{j=1}^{n}(-1)^{j+1}a_{1j}M_{1j}$$

Determinants are pretty nasty but we are fortunate as we really only need them for $n = 1, 2$ or 3.

9.2 Properties of the Determinant

1. Any matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same determinant, i.e. $\det(\mathbf{A}) = \det(\mathbf{A}^T)$.

   Note: This is useful since it implies that whenever we use rows, a similar behavior will result if we use columns. In particular we will see how row elementary operations are helpful in finding the determinant.

2. The determinant of a triangular matrix is the product of the entries on the diagonal, that is $\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix} = aei$

3. If we interchange two rows, the determinant of the new matrix is the opposite sign of the old one, that is

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

4. If we multiply one row by a constant, the determinant of the new matrix is the determinant of the old one multiplied by the constant, that is $\begin{pmatrix} a & b & c \\ d & e & f \\ \lambda g & \lambda h & \lambda i \end{pmatrix} = \lambda \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ In particular, if all the entries in one row are zero, then the determinant is zero.

5. If we add one row to another one multiplied by a constant, the determinant of the new matrix is the same as the old one, that is $\begin{pmatrix} a & b & c \\ d & e & f \\ \lambda a + g & \lambda b + h & \lambda c + i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

   Note that whenever you want to replace a row by something (through elementary operations), do not multiply the row itself by a constant. Otherwise, it is easy to make errors, see property 4

6. det($\mathbf{AB}$)=det($\mathbf{A}$)det($\mathbf{B}$)

7. $\mathbf{A}$ is invertible if and only if det($\mathbf{A}$) $\neq 0$. Note in that case det($\mathbf{A}^{-1}$)=1/det($\mathbf{A}$)

While determinants can be useful in geometry and theory they are complex and quite difficult to handle. Our last result is for completeness and links matrix inverses with determinants.

Recall that the $n \times n$ matrix $\mathbf{A}$ does not have an inverse when det($\mathbf{A}$)=0. However the connection between determinants and matrices is more complex. Suppose we define a new matrix, the adjoint of $\mathbf{A}$ say adj($\mathbf{A}$) as

$$\text{adj}\mathbf{A} = \left((-1)^{i+1} M_{ij}\right)^T = \begin{pmatrix} M_{11} & -M_{12} & \cdots & (-1)^{n+1} M_{1,n} \\ -M_{21} & M_{22} & \cdots & (-1)^{n+2} M_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^{n+1} M_{n1} & (-1)^{n+2} M_{n2} & \cdots & (-1)^{2n} M_{nn} \end{pmatrix}^T$$

Here the M_{ij} are just the minors defined above.

So if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & \cdots \end{pmatrix}$ then adj($\mathbf{A}$)= $\begin{pmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix}$

Why is anyone interested in the adjoint? The main reason is

$$\mathbf{A}^{-1} = \frac{\text{adj}\mathbf{A}}{\det(\mathbf{A})}$$

Of course you would have to have a very special reason to compute an inverse this way.

9.2.1 Cramer's Rule

Suppose we have the set of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

and let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then Cramer' s rule states that

$$x = \frac{1}{D}\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$y = \frac{1}{D}\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$z = \frac{1}{D}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

There is even a more general case. Suppose we have

$$\mathbf{A}\mathbf{x} = \mathbf{d}$$

where $\mathbf{x}^T = (x_1, x_2, \ldots, x_n)$ and $\mathbf{d}^T = (d_1, d_2, \ldots, d_n)$. Let D =det($\mathbf{A}$).

Then

$$x_k = \frac{1}{D}\begin{pmatrix} a_{11} & \cdots & a_{1(k-1)} & d_1 & a_{1(k+1)} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n(k-1)} & d_n & a_{n(k+1)} & \cdots & a_{nn} \end{pmatrix}$$

While this is a nice formula you would have to be mad to use it to solve equations since the best way of evaluating big determinants is by row reduction, and this gives solutions directly.

Exercises

1. Evaluate $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 2 & 4 & 3 \\ 3 & 6 & 5 \\ 2 & 5 & 2 \end{vmatrix}$

3. Evaluate $\begin{vmatrix} x & 1 & 2 \\ x^2 & 2x+1 & x^3 \\ 0 & 3x-2 & 2 \end{vmatrix}$

4. If $\mathbf{A} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & 14 \end{pmatrix}$ show that

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & d \end{vmatrix}$$